Percolation in restricted geometries and conformal invariance

1990 J. Phys. A: Math. Gen. 23 L153

(http://iopscience.iop.org/0305-4470/23/4/005)

J. Phys. A: Math. Gen. **23** (1990) L153-L157. Printed in the UK

LETTER TO THE EDITOR

Percolation in restricted geometries and conformal invariance

T Wolf, R Blender† and W Dietrich

Fakultät für Physik, Universität Konstanz, D-7750 Konstanz, Federal Republic of Germany

Received 25 September 1989

Abstract. We determine numerically the pair connectivity for two-dimensional percolation systems subject to different boundary geometries. Our results provide a sensitive and efficient test of the concept of conformal invariance and confirm its validity even for relatively small lattices. The influence of boundaries on electrical conduction in percolative networks is also addressed briefly.

The principle of conformal invariance of statistical systems at criticality has considerably improved our understanding of critical phenomena in restricted geometries (for a review see Cardy 1987). In this context it allows us to predict critical effects in a certain boundary geometry by conformal mapping of known results in a simpler geometry. Clearly, by the richness of analytic functions, this method becomes most powerful in two dimensions, and a variety of explicit examples of transforming correlation functions or order-parameter profiles has been worked out (see e.g. Cardy 1984a, Burkhardt and Eisenriegler 1985, Peschel and Truong 1987). Results of conformal invariance have also been verified by comparison against numerical simulations of 2D spin systems in disc and rectangular geometries (Badke *et al* 1985, Bartelt and Einstein 1986).

In this letter we present a numerical study of percolation on 2D lattices with different boundary shapes. The percolation model is equivalent to the q-state Potts model in the limit $q \to 1$ (Wu 1982 and references therein) and conformal invariance is expected to hold for the pair connectivity function $\Gamma(z_1, z_2)$ which is defined as the probability that two sites z_1 and z_2 are both occupied and belong to the same cluster. Under a conformal mapping $z \to w(z)$ of the complex z plane the function Γ should therefore transform as

$$\Gamma(z_1, z_2) = |w'(z_1)|^{x_1} |w'(z_2)|^{x_2} \Gamma(w_1, w_2). \tag{1}$$

Here $\Gamma(w_1, w_2)$ denotes the pair connectivity in the transformed geometry. The scaling dimension x_i $(i = 1, 2)$ corresponding to a point z_i in the bulk is related to the conventional bulk exponents $(d = 2)$ by $x_i \equiv x_b = \eta/2$ where $\eta = \beta/\nu$. In the present case of 2D percolation, $\beta = 5/36$ and $\nu = 4/3$ (see e.g. Stauffer 1985). On the other hand, for surface points z_i we have $x_i \equiv x_s = \eta_{\|}/2$ (Binder 1983). According to Cardy

† Present address: Institut für Meteorologie, Freie Universität Berlin, D-1000 Berlin 33, Federal Republic of Germany.

(1984b),

$$\eta_{\parallel}=2/(3\nu-1) \tag{2}$$

which leads to $\eta_{\parallel}=2/3$.

Our main objective is to calculate the pair connectivity for different geometries and to compare it with the predictions implied by (1) and (2). For that purpose we generate percolation clusters by using the Leath algorithm (Leath 1976) which allows us to examine the effects of boundaries in a very efficient way. Hence, in comparison with spin models, the percolation model is particularly convenient for numerical studies in this context.

According to the Leath algorithm, clusters are grown starting from an occupied centre z_1 in an otherwise empty lattice. Empty nearest-neighbour sites of an occupied site which are not forbidden are transformed with probability p into occupied sites and with probability $1-p$ into forbidden sites. The distribution of percolation clusters of occupied sites at critically is obtained by setting p equal to the threshold concentration p_c and applying the growth procedure repeatedly. Boundaries are naturally incorporated by defining sites beyond the boundary as forbidden sites from the outset. The function $\Gamma(z_1, z_2)$ is then obtained as the relative number of growth processes reaching z_2. In our procedure here we are actually dealing with bond percolation rather than site percolation, because we are interested in addition in combining results for $\Gamma(z_1, z_2)$ with the problem of electrical conduction in bounded geometries. The necessary adaption of the Leath method to bond percolation is straightforward. Boundaries are incorporated in this case by taking bonds across the boundary as non-conducting.

In a first step we consider two points on the surface of a half-plane. Figure 1 shows results for the connectivity function which are obtained by generating typically 10^4 to 5×10^4 clusters for each value of distance $z=|z_1-z_2|$. Clearly, the data are well described by a power law $\Gamma\sim z^{-\eta_{\parallel}}$. The exponent $\eta_{\parallel}=0.66\pm0.01$ determined numerically is in good agreement with (2).

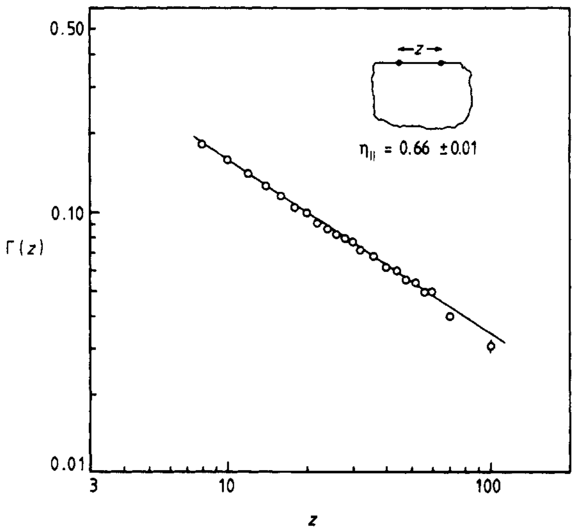

Figure 1. Simulation results for the pair connectivity Γ along the surface of a half-plane. Error bars, unless shown explicitly, are smaller than the size of data points. The fit by the straight line yields a slope $\eta_{\parallel}=0.66\pm0.01$.

Now we turn to specific testing examples of equation (1). Let us consider the geometry depicted in figure 2, where the boundary consists of straight line segments forming two corners z_1 and z_2 with an angle α. By equation (1), each corner contributes an exponent x_α, which can be determined from the map $w = z^{\alpha/\pi}$ (Cardy 1984b),

$$x_\alpha = (\pi/\alpha)\eta_\parallel/2. \tag{3}$$

Using a square lattice we first take $\alpha = \pi/2$ and calculate the pair connectivity $\Gamma(z_1, z_2)$ as described above. Results in the range $5 \leqslant |z_1 - z_2| \leqslant 10^2$ of distances (in units of the lattice constant) between the corners follow a power law with a quality similar to that observed in figure 1. The exponent, which we deduce, is $x_{\pi/2} = 0.65 \pm 0.01$, in accord with (3). Using boundaries parallel to the diagonal of the square lattice allows us to treat $\alpha = 3\pi/4$. In addition we use triangular lattices and finally obtain exponents x_α for a set of angles α, as seen in figure 2. Within the statistical uncertanties, our data, plotted against π/α, follow the straight line suggested by (3). We have also tested combinations of different angles α_1 and α_2 at the two corners and again find excellent agreement with the predictions from (1).

Next we assume

$$w(z) = \frac{2L}{\pi} \sin^{-1} z \tag{4}$$

which maps the upper half z plane $z = x + \mathrm{i}y$, $y \geqslant 0$, onto the semi-infinite strip of width $2L$, $w = u + \mathrm{i}v$, $-L \leq u \leq L$, $v > 0$. Surface points $z_1 = x_1 > 1$, $z_2 = -x_1$ transform into $w_1 = L + \mathrm{i}v$, $w_2 = -L + \mathrm{i}v$. Equation (1) in turn allows us to obtain the connectivity across the strip as a function of distance v from the corners,

$$\Gamma(2L, v) \sim \left(\frac{\pi}{2L}\right)^{\eta_\parallel} \left| \tanh\left(\frac{\pi v}{2L}\right)\right|^{\eta_\parallel}. \tag{5}$$

In the limit $v \gg L$ we obtain the result $\Gamma \sim \Gamma^{-\eta_\parallel}$ for the connectivity across an infinitely long strip, whereas in the opposite limit ($v \ll L$) we obtain $\Gamma \sim \Gamma^{-2\eta_\parallel}$, as expected from equation (3) with $\alpha = \pi/2$.

Figure 3 shows a test of the crossover behaviour (5) between those two limits. Plotting $\Gamma(2L, v)(2L)^{\eta_\parallel}$ against $v/2L$ we find that data collected for several values of L and v tend to collapse on a single curve in agreement with (5). The asymptote for large v has been determined independently from the L-dependent connectivity function

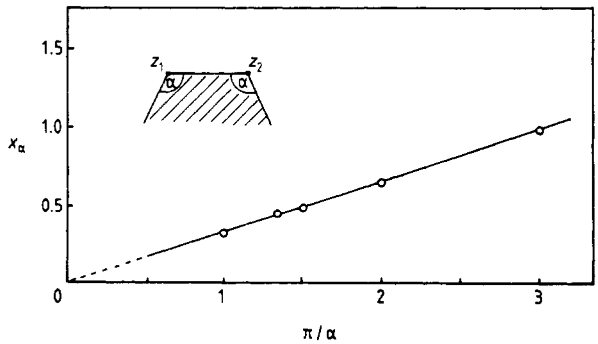

Figure 2. Comparison of exponents x_α obtained from simulations (circles) with the prediction based on conformal invariance, cf equation (3). Error bars correspond to the size of data points.

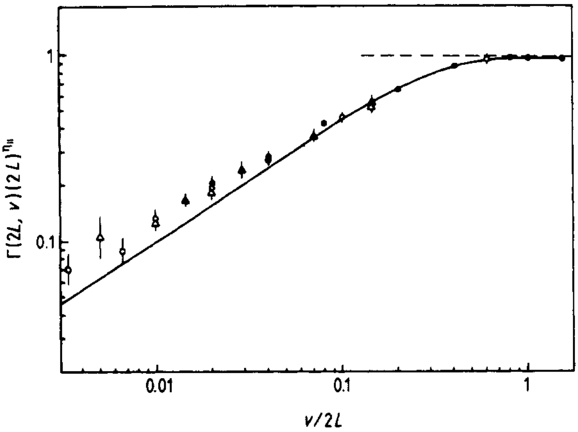

Figure 3. Pair connectivity across a semi-infinite strip of width $2L$ at a distance v from the corners. Data points are from simulations for $L = 50$ (●), 70 (▲), 100 (○), 200 (△), 300 (□). The scaling function according to equation (5) is represented by the full curve. The broken curve indicates the asymptote calculated for large v.

across an infinite strip. It is worth noting that numerical data even for rather small v are still well represented by (5) although the basic formula (1) refers to a continuum situation. Deviations form the full curve in figure 3 due to discreteness effects become noticable only for $v \lesssim 4$. Numerical calculations for $v = 0$ of course cannot be compared with (5). On a discrete lattice the connectivity for $v = 0$ is finite; see the case $\alpha = \pi/2$ in connection with (3).

In summary, we have shown that percolation is a very convenient model to provide numerical tests of conformal invariance with high accuracy. As a physical application of these concepts let us consider the problem of boundary effects with respect to electrical conduction. As a specific example we consider the resistance between point contacts at the two corners of a semi-infinite strip of width L. By $R(L)$ we denote the resistance averaged over conducting clusters subject to the geometry considered. Then we expect that

$$R(L) \sim L^{\zeta} \tag{6}$$

where the exponent ζ should not be affected by the presence of boundaries. In other words, $\zeta = t/\nu \simeq 0.98$ for $d = 2$ (Havlin and Ben-Avraham 1987 and references therein), where t denotes the standard conductivity exponent. To discuss this, let us employ the 'nodes-links-blobs' picture (Stanley 1977) of a cluster originally grown in the unrestricted two-dimensional plane. Introducing boundaries will remove a certain set of bonds in the conducting backbone. By the definition of $R(L)$, all configurations are discarded where the two contacts become disconnected. The remaining modification of the original 'blobs' should represent a minor disturbance which changes the prefactor in (6) but not the exponent. We have verified this by explicit transfer-matrix calculations (Derrida and Vannimenus 1982), taking into account the point contact geometry described above (Wolf 1989). From this we conclude that the averaged electrical conductance between two points in a percolation system, containing both conducting and non-conducting clusters, factorises according to $\bar{\Sigma} = \Gamma(L)R^{-1}(L)$. The first factor accounts for the probability of the two points to be connected and is geometry dependent whereas the second factor is essentially independent of geometry.

We acknowledge useful discussions with I Peschel, H E Roman and R B Stinchcombe. This work was supported in part by the Deutsche Forschungsgemeinschaft, SFB 306.

References

Badke R, Rittenberg V and Ruegg H 1985 *J. Phys. A: Math. Gen.* **18** L867
Bartelt N C and Einstein T L 1986 *J. Phys. A: Math. Gen.* **19** 1429
Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic)
Burkhardt T W and Eisenriegler E 1985 *J. Phys. A: Math. Gen.* **18** L83
Cardy J L 1984a *J. Phys. A: Math. Gen.* **17** L385
—— 1984b *Nucl. Phys.* B **240** 514
—— 1987 *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (London: Academic)
Derrida B and Vannimenus J 1982 *J. Phys. A: Math. Gen.* **15** L557
Havlin S and Ben-Avraham D 1987 *Adv. Phys.* **36** 695
Leath P L 1976 *Phys. Rev.* B **14** 5046
Peschel I and Truong T T 1987 *Z. Phys.* B **69** 385
Stanley H E 1977 *J. Phys. A: Math. Gen.* **10** L211
Stauffer D 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)
Wolf T 1989 *Diplomarbeit* Universität Konstanz, unpublished
Wu F Y 1982 *Rev. Mod. Phys.* **54** 235